J Glob Optim (2007) 39:365–392
DOI 10.1007/s10898-007-9142-4

ORIGINAL PAPER

A multiplicity theorem for the Neumann p-Laplacian with an asymmetric nonsmooth potential

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Received: 28 September 2006 / Accepted: 17 January 2007 / Published online: 21 February 2007

Abstract We consider a nonlinear Neumann problem driven by the p-Laplacian differential operator with a nonsmooth potential (hemivariational inequality). By combining variational with degree theoretic techniques, we prove a multiplicity theorem. In the process, we also prove a result of independent interest relating $W_n^{1,p}$ and C_n^1 local minimizers, of a nonsmooth locally Lipschitz functional.

Keywords Neumann problem · p-Laplacian · Degree theory · Local minimizer · Lagrange multiplier rule · Nonsmooth potential

2000 AMS subject classification 35J25 · 35J70

1 Introduction

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary ∂Z. In this paper, we study the following nonlinear Neumann problem with a nonsmooth potential (hemivariational inequality):

$$\begin{aligned} -\mathrm{div}(|Dx(z)|^{p-2}Dx(z)) \in \partial j(z,x(z)), &\quad \text{a.e. on } Z, \\ \frac{\partial x}{\partial n_p} = 0, &\quad \text{on } \partial Z,\ 2 \leq p < +\infty. \end{aligned} \tag{1}$$

Here $j(z,x)$ is a measurable potential function on $Z \times \mathbb{R}$, which is only locally Lipschitz and in general nonsmooth in the $x \in \mathbb{R}$ variable. By $\partial j(z,x)$ we denote the

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generalized subdifferential of the locally Lipschitz function $x \to j(z,x)$. Also $\frac{\partial x}{\partial n_p} = |Dx|^{p-2}(Dx,n)_{\mathbb{R}^N}$ with n being the outward unit normal on ∂Z. Our goal is to prove the existence of multiple nontrivial solutions for problem (1).

While for the Dirichlet problem of the p-Laplacian, there have been several multiplicity results, the case of the Neumann problem is lagging behind. Of the existing works in the literature, the majority deal with problems in which the potential function is smooth (i.e., $j(z,\cdot) \in C^1(\mathbb{R})$). We mention the papers of Anello [4] and Ricceri [27], who for the case $p \geq 2$ established the existence of infinitely many solutions for certain nonlinear eigenvalue problems, using symmetry conditions on the nonlinearity. Bonanno and Candito [6] considered a problem similar to that of Ricceri [27] and using a "three critical point" theorem of Ricceri [26], proved the existence of three solutions for the nonlinear eigenvalue problem. Binding et al. [5], considered problems in which the right-hand side nonlinearity is $\lambda a(z)|x|^{p-2}x + b(z)|x|^{p^*-2}x + h(z)$, with $a,\, b \in L^\infty(Z)_+, h \in L^\infty(Z)$ and proved the existence of one or two positive solutions. Faraci [13] and Wu and Tan [30], studied problems where $p > N$ and exploited the fact that in this case the Sobolev space $W^{1,p}(Z)$ is embedded compactly in $C(\overline{Z})$. Problems with a nonsmooth potential, were studied by Marano and Molica Bisci [24], Papageorgiou and Smyrlis [25] and Filippakis et al. [14]. Marano and Molica Bisci [24] use the Ambrosetti–Rabinowitz condition and the local linking theorem. Papageorgiou and Smyrlis [25] and Filippakis et al. [14], imposing symmetry conditions on the potential, prove the existence of infinitely many solutions. In all the above works (smooth and nonsmooth alike), the nonlinearity (single-valued and multivalued), exhibits a symmetric behavior as we approach $+\infty$ and $-\infty$. This is no longer the case with our work. The multivalued nonlinearity $\partial j(z,x)$ is asymmetric near $+\infty$ and $-\infty$ and the "generalized slope" $\left\{\frac{u}{|x|^{p-2}x}\right\}_{u\in\partial j(z,x)}$ crosses the principal eigenvalue $\lambda_0 = 0$ as we move from $-\infty$ to $+\infty$. Moreover, our approach here is different from all the aforementioned papers and it is based on a combination of variational and degree theoretic techniques.

2 Mathematical background

Let X be a Banach space and X^* its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*,X). For a locally Lipschitz function $\varphi\colon X \to \mathbb{R}$ the generalized directional derivative $\varphi^0(x;h)$ of φ at $x \in X$ in the direction $h \in X$, is defined by

$$\varphi^0(x;h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x'+\lambda h) - \varphi(x')}{\lambda}\,.$$

It is easy to check that $\varphi^0(x;\cdot)$ is sublinear continuous. So from the Hahn–Banach theorem it follows that $\varphi^0(x;\cdot)$ is the support function of a nonempty, w-compact, convex set $\partial\varphi(x) \subseteq X^*$ defined by

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*,h\rangle \leq \varphi^0(x;h)\ \forall\, h \in X\}\,.$$

The multifunction $\partial\varphi\colon X \to 2^{X^*}\setminus\{\emptyset\}$ is the generalized subdifferential of φ. If $\varphi \in C^1(X)$, then it is locally Lipschitz and $\partial\varphi(x) = \{\varphi'(x)\}$ for all $x \in X$. Also if

φ: $X \to \mathbb{R}$ is continuous convex, then φ is locally Lipschitz and the generalized subdifferential coincides with the subdifferential in the sense of convex analysis, defined by

$$\partial_C \varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \leq \varphi(x+h) - \varphi(x)\ \forall\, h \in X\}.$$

A multifunction G: $X \to 2^{X^*} \setminus \{\emptyset\}$ is said to be upper semicontinuous (u.s.c. for short), if for every closed set $C \subseteq X^*$

$$G^-(C) = \{x \in X : G(x) \cap C \neq \emptyset\}$$

is closed in X. We say that a multifunction G: $X \to 2^{X^*} \setminus \{\emptyset\}$ belongs to the class (P) if it is u.s.c., has closed and convex values and for every $B \subseteq X$ bounded, the set

$$G(B) = \cup_{x \in B} G(x)$$

is relatively compact in X^*.
If G: $D \subseteq X \to 2^{X^*} \setminus \{\emptyset\}$ is an u.s.c. multifunction with closed and convex values, then for every $\varepsilon > 0$, we can find a continuous map g_ε: $D \to X^*$ such that

$$g_\varepsilon(x) \in G\left((x + B_\varepsilon) \cap D)\right) + B^*_\varepsilon \quad \text{for all } x \in D \text{ and } g_\varepsilon(D) \subseteq \overline{conv}\, G(D).$$

Here $B_\varepsilon = \{x \in X : \|x\| < \varepsilon\}$ and $B^*_\varepsilon = \{x^* \in X^* : \|x^*\| < \varepsilon\}$ (see Hu and Papageorgiou [19], p. 106). Note that if G belongs to the class (P), then the continuous approximate selector g_ε is a compact map.

Using the continuous approximate selector, we can define a degree map for certain multivalued perturbations of $(S)_+$-operators. To this end, let X be a reflexive Banach space. By the Troyanski renorming theorem (see for example Gasinski and Papageorgiou [16], p. 911), we can equivalently renorm X so that both X and X^* are locally uniformly convex and have Fréchet differentiable norms. So in what follows, we assume without loss of generality that both Banach spaces X and X^* are locally uniformly convex. Then the duality map of X, $\mathcal{F}$: $X \to X^*$ defined by

$$\mathcal{F}(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}$$

is a homeomorphism.

Let A: $X \to X^*$ be a single-valued, nonlinear operator, which is defined on all of X. We say that A is of type $(S)_+$, if for every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $x_n \rightharpoonup x$ in X and $\limsup_{n\to+\infty} \langle A(x_n), x_n - x\rangle \leq 0$, one has $x_n \to x$ in X.
Let $U \subseteq X$ be a bounded open set and A: $\overline{U} \to X^*$ a bounded, demicontinuous operator of type $(S)_+$. Let $\{X_\alpha\}_{\alpha\in J}$ be the collection of all finite dimensional subspaces of X and let A_α be the Galerkin approximation of A with respect to X_α, that is

$$\langle A_\alpha(x), y\rangle_{X_\alpha} = \langle A(x), y\rangle \quad \text{for all } x \in \overline{U} \cap X_\alpha, \text{ for all } y \in X_\alpha.$$

Then for $x^* \notin A(\partial U)$, the degree map $d_{(S)_+}(A,\, U,\, x^*)$ is defined by

$$d_{(S)_+}(A,\, U,\, x^*) = d_B(A_\alpha,\, U \cap X_\alpha,\, x^*)$$

for X_α large enough (in the sense of inclusion), where d_B stands for the classical Brouwer's degree map. If X is a separable reflexive Banach space and the nonlinear operator A is bounded (namely it maps bounded sets in X to bounded sets in X^*), then we can use only a countable subfamily $\{X_n\}_{n\geq 1}$ of $\{X_\alpha\}_{\alpha\in J}$ such that $X = \overline{\cup_{n\geq 1} X_n}$. For further details on the degree map $d_{(S)_+}$, we refer to Skrypnik [28] and Browder [8].

Suppose $G\colon X \to 2^{X^*} \setminus \{\emptyset\}$ is a multifunction in the class (P). For every $x^* \notin (A+G)(\partial U)$, the degree map $\widehat{d}(A+G, U, x^*)$, is defined by

$$\widehat{d}(A+G,\, U,\, x^*) = d_{(S)_+}(A+g_\varepsilon,\, U,\, x^*)$$

for $\varepsilon > 0$ small. Here $g_\varepsilon\colon \overline{U} \to X^*$ is the continuous ε-approximate selector of G as described above. Note that g_ε is compact and so $x \to A(x) + g_\varepsilon(x)$ is of type $(S)_+$. More on this degree map, can be found in Hu and Papageorgiou [18] (see also Hu and Papageorgiou [19], Sect. 4.4).
One of the fundamental properties of any degree map, is the "homotopy invariance" property. So next we describe the admissible homotopies for the operator A and the multifunction G.

Definition 2.1

(a) A one parameter family $\{A_t\}_{t\in[0,1]}$ of bounded operators from $\overline{U}$ into X^*, is said to be an "$(S)_+$-homotopy", if for any $\{x_n\}_{n\geq 1} \subseteq \overline{U}$ such that $x_n \rightharpoonup x$ and for any $\{t_n\}_{n\geq 1} \subseteq [0,1]$ with $t_n \to t$, for which

$$\limsup_{n\to+\infty} \langle A_{t_n}(x_n),\, x_n - x\rangle \leq 0$$

we have $x_n \to x$ in X and $A_{t_n}(x_n) \rightharpoonup A_t(x)$ in X^*.
(b) A one parameter family $\{G_t\}_{t\in[0,1]}$ of multifunctions $G_t\colon \overline{U} \to 2^{X^*} \setminus \{\emptyset\}$ is said to be a "homotopy of class (P)", if $(t,x) \to G_t(x)$ is u.s.c. from $[0,1] \times \overline{U}$ into $2^{X^*} \setminus \{\emptyset\}$, for every $(t,x) \in [0,1] \times \overline{U}$ the set $G_t(x)$ is closed, convex and

$$\overline{\cup\{G_t(x) : t \in [0,1],\, x \in \overline{U}\}}$$

is compact in X^*.

Then the homotopy invariance property for the degree map $\widehat{d}$, reads as follows: "If $\{A_t\}_{t\in[0,1]}$ is an $(S)_+$-homotopy, $\{G_t\}_{t\in[0,1]}$ is a homotopy of class (P) and $x^*\colon [0,1] \to X^*$ is a continuous map such that

$$x_t^* \notin (A_t + G_t)(\partial U) \quad \text{for all } t \in [0,1],$$

then $\widehat{d}(A_t + G_t,\, U,\, x_t^*)$ is independent of $t \in [0,1]$."
The two degree maps $d_{(S)_+}$, $\widehat{d}$ have all the usual properties (namely homotopy invariance, solution property, domain additivity, excision property). Note that in this case the normalization property, has the form

$$d_{(S)_+}(\mathcal{F},\, U,\, x^*) = 1 \quad \text{for all } x^* \in \mathcal{F}(U).$$

Here $\mathcal{F}\colon X \to X^*$ is the duality map of X, which is bounded and $(S)_+$.
We denote by $\|\cdot\|_p$ the usual norm on $L^p(Z)$ and by $\|\cdot\|$ that on $W^{1,p}(Z)$. Finally, we recall some basic facts about the spectrum of the negative p-Laplacian with Neumann boundary conditions. So let $m \in L^\infty(Z)_+, m \neq 0$ and consider the following nonlinear weighted (with weight m) eigenvalue problem:

$$\begin{aligned} &-\mathrm{div}(|Dx(z)|^{p-2}Dx(z)) = \widehat{\lambda} m(z)|x(z)|^{p-2}x(z) \quad \text{a.e. on } Z,\\ &\frac{\partial x}{\partial n_p} = 0 \quad \text{on } \partial Z,\ 1 < p < +\infty,\ \widehat{\lambda} \in \mathbb{R}. \end{aligned} \tag{2}$$

A $\widehat{\lambda} \in \mathbb{R}$ for which problem (2) has a nontrivial solution, is said to be an eigenvalue of $(-\Delta_p, W^{1,p}(Z), m)$ and the nontrivial solution is an eigenfunction corresponding to the eigenvalue $\widehat{\lambda}$. It is easy to see that a necessary condition for $\widehat{\lambda}$ to be an eigenvalue, is that $\widehat{\lambda} \geq 0$. Moreover, zero is an eigenvalue with corresponding eigenspace $\mathbb{R}$ (that is the space of constant functions). This eigenvalue is also isolated and admits the following variational characterization

$$0 = \widehat{\lambda}_0(m) = \inf\left[\frac{\|Dx\|_p^p}{\int_Z m|x|^p \mathrm{d}z} : x \in W^{1,p}(Z),\ x \neq 0\right]. \tag{3}$$

Clearly the constant functions realize the infimum in (3).

By the Liusternik–Schnirelmann theory, in addition to $\widehat{\lambda}_0(m) = 0$, we have a whole strictly increasing sequence $\{\widehat{\lambda}_k = \widehat{\lambda}_k(m)\}_{k\geq 0} \subseteq \mathbb{R}_+$ such that $\widehat{\lambda}_k \to +\infty$ as $k \to +\infty$, which are eigenvalues of $(-\Delta_p, W^{1,p}(Z), m)$. These are the so-called "variational eigenvalues" of $(-\Delta_p, W^{1,p}(Z), m)$.

If $p = 2$ (linear eigenvalue problem), then the variational eigenvalues are all the eigenvalues of $(-\Delta, H^1(Z), m)$. If $p \neq 2$ (nonlinear eigenvalue problem), we do not know if this is so. However, since $\widehat{\lambda}_0 = 0$ is isolated and the set $\sigma(p,m)$ of all the eigenvalues of $(-\Delta_p, W^{1,p}(Z), m)$ is closed, then

$$\widehat{\lambda}_1^* = \inf\left[\widehat{\lambda} : \widehat{\lambda} \in \sigma(p,m),\ \widehat{\lambda} > 0\right] \in \sigma(p,m)$$

and $\widehat{\lambda}_1^* = \widehat{\lambda}_1$.
So the first two eigenvalues of $(-\Delta_p, W^{1,p}(Z), m)$ coincide with the first two variational eigenvalues.

$$\text{Let } \varphi_m(x) = \int_Z m|x|^p \mathrm{d}z, \quad \psi_m(x) = \int_Z m|x|^p \mathrm{d}z + \|Dx\|_p^p \quad \text{for all } x \in W^{1,p}(Z),$$

$$S(\psi_m) = \left\{x \in W^{1,p}(Z) : \psi_m(x) = 1\right\},$$

$$\text{and } \mathcal{A}_k = \left\{C \subseteq S(\psi_m) : C \text{ is compact, symmetric and } \gamma(C) \geq k\right\}.$$

Here by $\gamma(\cdot)$ we denote the Krasnoselskii genus (see Gasinski and Papageorgiou [16], p. 679). We have

$$\frac{1}{\widehat{\lambda}_k(m) + 1} = \sup_{C\in\mathcal{A}_k} \min_{x\in C} \varphi_m(x) \text{ for all } k \in \{1, 2, \ldots\}. \tag{4}$$

If $m \equiv 1$, then we write $\lambda_k = \widehat{\lambda}_k$ for all $k \geq 0$.
Further details on these and related issues can be found in Lê [22] and in Gasinski and Papageorgiou [16].

3 One positive solution

The hypothesis on the nonsmooth potential $j(z,x)$ are the following:
$\underline{H(j)}$: j: $Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(z,0) = 0$ a.e. on Z, $\partial j(z,0) \subseteq \mathbb{R}_+$ a.e. on $\overline{Z}$ and

(1) for all $x \in \mathbb{R}$, $z \to j(z,x)$ is measurable;
(2) for almost all $z \in Z$, $x \to j(z,x)$ is locally Lipschitz;

(3) for every $r > 0$, there exists $a_r \in L^\infty(Z)_+$ such that for almost all $z \in Z$, all $|x| \le r$ and all $u \in \partial j(z,x)$, we have

$$|u| \le a_r(z)\,;$$

(4) there exists $\theta \in L^\infty(Z)$ such that $\theta(z) \le 0$ a.e. on Z with strict inequality on a set of positive measure and

$$\limsup_{x\to+\infty} \frac{u}{x^{p-1}} \le \theta(z)$$

uniformly for almost all $z \in Z$ and all $u \in \partial j(z,x)$ and also there exist $\theta_1, \theta_2 \in L^\infty(Z)_+$ such that $0 \le \theta_1(z) \le \theta_2(z) < \lambda_1$ a.e. on Z with the first inequality strict on a set of positive measure and

$$\theta_1(z) \le \liminf_{x\to-\infty} \frac{u}{|x|^{p-2}x} \le \limsup_{x\to-\infty} \frac{u}{x^{p-2}x} \le \theta_2(z)$$

uniformly for almost all $z \in Z$ and all $u \in \partial j(z,x)$;

(5) there exist $\eta_1, \eta_2 \in L^\infty(Z)$ such that

$$\eta_1(z) \le \eta_2(z) \le 0 \quad \text{a.e. on } Z$$

with the second inequality strict on a set of positive measure and

$$\eta_1(z) \le \liminf_{x\to 0} \frac{u}{|x|^{p-2}x} \le \limsup_{x\to 0} \frac{u}{|x|^{p-2}x} \le \eta_2(z)$$

uniformly for almost all $z \in Z$ and all $u \in \partial j(z,x)$;

(6) there exist $c_0 > 0$, $c > 0$ and $M_0 > 0$ such that

$$\int_Z j(z,c_0)\mathrm{d}z > 0\,,$$

$$-cx^{p-1} \le u \text{ for almost all } z \in Z,\ \text{all } x \ge 0 \text{ and all } u \in \partial j(z,x)$$
$$\text{and } u \le 0 \text{ for almost all } z \in Z,\ \text{all } x \ge M_0 \text{ and all } u \in \partial j(z,x).$$

Remark 3.1 Clearly hypothesis $H(j)$ (4) describes an asymmetric behavior of $\partial j(z,\cdot)$ at $+\infty$ and at $-\infty$. Near $+\infty$ we have partial interaction (nonuniform nonresonance) of the "generalized slope" $\left\{\frac{u}{x^{p-1}} : u \in \partial j(z,x)\right\}$ with the principal eigenvalue $\lambda_0 = 0$ from the left (i.e., from below), while near $-\infty$ the "generalized slope" remains in the spectral interval $[\lambda_0 = 0,\ \lambda_1]$, with nonuniform nonresonance with respect to $\lambda_0 = 0$ and uniform nonresonance with respect to λ_1. Near zero (see $H(j)$ (5)), we have a symmetric behavior of the generalized slope $\left\{\frac{u}{|x|^{p-2}x} : u \in \partial j(z,x)\right\}$. We have nonuniform nonresonance with respect to the principal eigenvalue $\lambda_0 = 0$. Observe that on the negative semiaxis, as we move from $-\infty$ to 0^-, we cross the principal eigenvalue $\lambda_0 = 0$ (crossing nonlinearity).

A simple nonsmooth, locally Lipschitz potential function $j(z,x)$ satisfying hypothesis $H(j)$, is the following:

$$j(z,x) = \begin{cases} \frac{\eta(z)}{p}|x|^p + \frac{\theta(z)-\eta(z)}{p} + \ln 2, & \text{if } x < -1, \\ \frac{\theta(z)}{p}|x|^p + |x|^p \ln(1+|x|), & \text{if } |x| \le 1, \\ \frac{\theta(z)}{p}|x|^p + \ln 2, & x > 1, \end{cases}$$

where $\theta(z) \le 0 \le \eta(z) < \lambda_1$ a.e. on $Z, \theta,\ \eta \neq 0$ and $-\int_Z \theta(z)dz < p\ln 2$.
In the analysis of problem (1), we will use the following two spaces:

$$W_n^{1,p}(Z) = \left\{ x \in W^{1,p}(Z) : x = \lim_{k\to\infty} x_k \text{ in } W^{1,p}(Z), x_k \in C^\infty(Z), \frac{\partial x_k}{\partial n} = 0 \quad \text{on } \partial Z \right\}$$

and

$$C_n^1(\overline{Z}) = \left\{ x \in C^1(\overline{Z}) : \frac{\partial x}{\partial n} = 0 \quad \text{on } \partial Z \right\}.$$

Both are Banach spaces with ordered cones given by

$$W_+ = \left\{ x \in W_n^{1,p}(Z) : x(z) \ge 0 \quad \text{a.e. on } Z \right\}$$

$$\text{and } C_+ = \left\{ x \in C_n^1(\overline{Z}) : x(z) \ge 0 \quad \text{for all } z \in \overline{Z} \right\}.$$

Moreover, we know that $\operatorname{int} C_+ \neq \emptyset$ and more precisely

$$\operatorname{int} C_+ = \{x \in C_+ : x(z) > 0 \quad \text{for all } z \in \overline{Z}\}.$$

In this section, we prove the existence of a solution $x_0 \in \operatorname{int} C_+$ for problem (1). To do this we will need a general result of independent interest about local $C_n^1(\overline{Z})$-minimizers of certain locally Lipschitz functionals. It extends earlier results of Brézis and Nirenberg [7] and Garcia Azorero et al. [15]. In Brézis and Nirenberg [7], $p = 2$ (semilinear case), the boundary conditions are Dirichlet and the potential function is smooth (i.e., $j(z,\cdot) \in C^1(\mathbb{R})$). In Garcia Azorero et al. [15], $p \neq 2$ (nonlinear case), but the boundary conditions and the potential functions are as in Brézis and Nirenberg [7]. So we introduce the following hypotheses:
$\underline{H(j_0)}$: j_0: $Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

(1) for all $x \in \mathbb{R}$, $z \to j_0(z,x)$ is measurable;
(2) for almost all $z \in Z$, $x \to j_0(z,x)$ is locally Lipschitz;
(3) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$, we have

$$|u| \le a_0(z) + c_0|x|^{r-1},$$

with $a_0 \in L^\infty(Z)_+$, $c_0 > 0$ and $1 < r < p^* = \begin{cases} \frac{Np}{N-p}, & \text{if } N > p\,, \\ +\infty, & \text{if } N \le p\,. \end{cases}$

We consider the functional φ_0: $W_n^{1,p}(Z) \to \mathbb{R}$ defined by

$$\varphi_0(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z j_0(z,x(z))\mathrm{d}z \quad \text{for all } x \in W_n^{1,p}(Z)\,.$$

We know that φ_0 is Lipschitz continuous on bounded sets, hence locally Lipschitz (see Clarke [11], p. 83).

Proposition 3.1 *If $x_0 \in W_n^{1,p}(Z)$ is a local $C_n^1(\overline{Z})$-minimizer of φ_0, i.e., there exists $\rho_1 > 0$ such that*

$$\varphi_0(x_0) \le \varphi_0(x_0 + h) \quad \textit{for all } h \in C_n^1(\overline{Z}), \quad \|h\|_{C_n^1(\overline{Z})} \le \rho_1$$

then $x_0 \in C^1_n(\overline{Z})$ *and is a local* $W^{1,p}_n(Z)$*-minimizer of* φ_0*, i.e., there exists* $\rho_2 > 0$ *such that*

$$\varphi_0(x_0) \le \varphi_0(x_0 + h) \quad \textit{for all } h \in W^{1,p}_n(Z), \quad \|h\| \le \rho_2.$$

Proof Let $h \in C^1_n(\overline{Z})$. If $\lambda > 0$ is small, then by hypothesis

$$\varphi_0(x_0) \le \varphi_0(x_0 + \lambda h)$$

and from this it follows that

$$0 \le \varphi_0^0(x_0; h) \quad \text{for all } h \in C^1_n(\overline{Z}). \tag{5}$$

Recall that $\varphi_0^0(x_0; \cdot)$ is continuous on $W^{1,p}_n(Z)$ and that $C^1_n(\overline{Z})$ is dense in $W^{1,p}_n(Z)$. So from (5), we infer that

$$0 \le \varphi_0^0(x_0; h) \quad \text{for all } h \in W^{1,p}_n(Z)$$

so

$$0 \in \partial\varphi_0(x_0). \tag{6}$$

In what follows by $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_0$ we denote, respectively, the duality brackets for the pairs $\left(W^{1,p}_n(Z)^*, W^{1,p}_n(Z)\right)$ and $\left(W^{-1,p'}(Z), W^{1,p}_0(Z)\right)$. Let A: $W^{1,p}_n(Z) \to W^{1,p}_n(Z)^*$ be the nonlinear operator defined by

$$\langle A(x), y \rangle = \int_Z |Dx|^{p-2}(Dx, Dy)_{\mathbb{R}^N} \mathrm{d}z\,.$$

From Clarke [10], p. 83, we know that any $v \in \partial\varphi_0(x_0)$ can be written as

$$v = A(x_0) - u_0$$

with $u_0 \in L^{r'}(Z)$ ($\frac{1}{r} + \frac{1}{r'} = 1$), $u_0(z) \in \partial j_0(z, x_0(z))$ a.e. on Z. So from (6), we have

$$A(x_0) = u_0 \quad \text{for some } u_0 \in L^{r'}(Z), \quad u_0(z) \in \partial j_0(z, x_0(z)) \quad \text{a.e. on } Z\,. \tag{7}$$

From the representation theorem for the elements of $W^{-1,p'}(Z) = W^{1,p}_0(Z)^*$ ($\frac{1}{p} + \frac{1}{p'} = 1$) (see for example Adams [1], p. 50), we know that

$$-\operatorname{div}\left(|D(x_0)|^{p-2}D(x_0)\right) \in W^{-1,p'}(Z)\,. \tag{8}$$

On (7) we act with $v \in C^1_c(Z)$ and obtain

$$\langle A(x_0), v \rangle = \int_Z |Dx_0|^{p-2}(Dx_0, Dv)_{\mathbb{R}^N} \mathrm{d}z = \int_Z u_0 v \, \mathrm{d}z\,. \tag{9}$$

Then from the definition of the distributional derivative, (8) and (9), we have

$$\langle A(x_0), v \rangle = \langle -\operatorname{div}(|Dx_0|^{p-2}Dx_0), v \rangle_0$$

so (9) forces

$$\langle -\operatorname{div}(|Dx_0|^{p-2}Dx_0), v \rangle_0 = \int_Z u_0 v \, \mathrm{d}z = \langle u_0, v \rangle_0 \quad \text{for all } v \in C^1_c(Z). \tag{10}$$

Since $C_c^1(Z)$ is dense in $W_0^{1,p}(Z)$, from (10) it follows that

$$-\operatorname{div}(|Dx_0(z)|^{p-2}Dx_0(z)) = u_0(z) \quad \text{a.e. on } Z. \tag{11}$$

Invoking Green's identity for quasi-linear operators (see Casas and Fernandez [9] and Kenmochi [20]), we have

$$\begin{aligned}\int_Z [\operatorname{div}(|Dx_0|^{p-2}Dx_0)]v\,\mathrm{d}z &+ \int_Z |Dx_0|^{p-2}(Dx_0, Dv)_{\mathbb{R}^N}\mathrm{d}z \\ &= \left\langle \frac{\partial x_0}{\partial n_p}, \gamma_0(v)\right\rangle_{\partial Z} \qquad \text{for all } v \in W^{1,p}(Z)\,.\end{aligned} \tag{12}$$

Here by $\langle\cdot,\cdot\rangle_{\partial Z}$ we denote the duality brackets for the pair $\left(W^{-\frac{1}{p'},p'}(\partial Z),\ W^{\frac{1}{p'},p}(\partial Z)\right)$ and γ_0 is the trace map on $W^{1,p}(Z)$. Using (11) in (12), we obtain

$$-\int_Z u_0 v\,\mathrm{d}z + \int_Z |Dx_0|^{p-2}(Dx_0, Dv)_{\mathbb{R}^N}\mathrm{d}z = \left\langle \frac{\partial x_0}{\partial n_p}, \gamma_0(v)\right\rangle_{\partial Z};$$

now, (9) and the previous equality yield

$$\left\langle \frac{\partial x_0}{\partial n_p}, \gamma_0(v)\right\rangle_{\partial Z} = 0 \quad \text{for all } v \in W^{1,p}(Z)\,. \tag{13}$$

Recall that $\gamma_0(W^{1,p}(Z)) = W^{1/p',p}(\partial Z)$. So from (13), we infer that

$$\frac{\partial x_0}{\partial n_p} = 0 \text{ in } W^{-1/p',p'}(\partial Z). \tag{14}$$

From Theorem 7.1, p. 286, of Ladyzhenskaya and Uraltseva [21], we have $x_0 \in L^\infty(Z)$. Then from Theorem 2 of Lieberman [23] implies $x_0 \in C_n^{1,\beta}(\overline{Z})$ for some $0 < \beta < 1$. So in (14), the equality is interpreted pointwise on ∂Z, i.e., $x_0 \in C_n^1(\overline{Z})$.

Suppose the proposition was not true. Since φ_0 is weakly lower semicontinuous and the set $\overline{B}_\varepsilon = \{h \in W_n^{1,p}(Z) : \|h\| \le \varepsilon\}$ is w-compact, by the Weierstrass theorem, for any $\varepsilon > 0$ we can find $h_\varepsilon \in \overline{B}_\varepsilon$ such that

$$\varphi_0(x_0 + h_\varepsilon) = \min[\varphi_0(x_0 + h) : h \in \overline{B}_\varepsilon] < \varphi_0(x_0). \tag{15}$$

From the nonsmooth Lagrange multiplier rule of Clarke [10], we can find $\lambda_\varepsilon \le 0$ such that

$$\lambda_\varepsilon \theta'_\varepsilon(h_\varepsilon) \in \partial\varphi_0(x_0 + h_\varepsilon),$$

where $\theta_\varepsilon(h) = \frac{1}{p}(\|h\|^p - \varepsilon^p)$. It follows that

$$A(x_0 + h_\varepsilon) - u_\varepsilon = \lambda_\varepsilon A(h_\varepsilon) + \lambda_\varepsilon K(h_\varepsilon), \tag{16}$$

where $u_\varepsilon \in L^{r'}(Z)$, $u_\varepsilon(z) \in \partial j_0(z, (x_0 + h_\varepsilon)(z))$ a.e. on Z and K: $L^p(Z) \to L^{p'}(Z)$ is defined by $K(h)(\cdot) = |h(\cdot)|^{p-2}h(\cdot)$. Evidently, $K_{|W_n^{1,p}(Z)}$ is completely continuous into $L^{p'}(Z) \subseteq W_n^{1,p}(Z)^*$. From (16) and (7), we have

$$A(x_0 + h_\varepsilon) - A(x_0) - \lambda_\varepsilon A(h_\varepsilon) = u_\varepsilon - u_0 + \lambda_\varepsilon K(h_\varepsilon).$$

From this operator equation, as before, we deduce that

$$\begin{aligned}&-\Delta_p(x_0+h_\varepsilon)(z)+\Delta_p x_0(z)+\lambda_\varepsilon\Delta_p h_\varepsilon(z)\\&\quad=u_\varepsilon(z)-u_0(z)+\lambda_\varepsilon|h_\varepsilon(z)|^{p-2}h_\varepsilon(z)\quad\text{a.e. on } Z.\end{aligned}\tag{17}$$

We introduce the function V: $Z\times\mathbb{R}^N\to\mathbb{R}^N$ defined by

$$V(z,\xi)=|Dx_0(z)+\xi|^{p-2}(Dx_0(z)+\xi)-|Dx_0(z)|^{p-2}Dx_0(z)-\lambda_\varepsilon|\xi|^{p-2}\xi.$$

Evidently $V(z,\xi)$ is a Carathéodory function (i.e., it is measurable in $z\in Z$ and continuous in $\xi\in\mathbb{R}^N$, hence jointly measurable on $Z\times\mathbb{R}^N$) and exhibits a $(p-1)$-polynomial growth in the $\xi\in\mathbb{R}^N$ variable. We rewrite (17) as follows

$$-\operatorname{div}V(z,h_\varepsilon(z))=u_\varepsilon(z)-u_0(z)+\lambda_\varepsilon|h_\varepsilon(z)|^{p-2}h_\varepsilon(z)\quad\text{a.e. on } Z.\tag{18}$$

As earlier, via Green's identity for quasi-linear operators (see Casas and Fernandez [9] and Kenmochi [20]), we have

$$\frac{\partial h_\varepsilon}{\partial n_p}(z)=0\quad\text{for all } z\in\partial Z.\tag{19}$$

Since $2\le p<+\infty$ and $\lambda_\varepsilon\le 0$, for almost all $z\in Z$ and all $\xi\in\mathbb{R}^N$

$$(V(z,\xi),\xi)_{\mathbb{R}^N}\ge\widehat{c}|\xi|^p-\lambda_\varepsilon|\xi|^p\ge\widehat{c}|\xi|^p.$$

So from (18), (19), Theorem 7.1, p. 286 of Ladyzhenskaya and Uraltseva [21] and Theorem 2 of Lieberman [23], we can find $\beta_0\in(0,1)$ and $\gamma_0>0$, both independent of $\varepsilon\in(0,1]$ and λ_ε such that

$$h_\varepsilon\in C_n^{1,\beta_0}(\overline{Z})\quad\text{and } \|h_\varepsilon\|_{C_n^{1,\beta_0}(\overline{Z})}\le\gamma_0.\tag{20}$$

Now let $\varepsilon_n\downarrow 0$ and set $h_n=h_{\varepsilon_n}$. Since $C_n^{1,\beta_0}(\overline{Z})$ is compactly embedded in $C_n^1(\overline{Z})$, we may assume that

$$h_n\to\widehat{h}\ \text{ in }\ C_n^1(\overline{Z})\ \text{ as }\ n\to+\infty.$$

On the other hand, $\|h_n\|\le\varepsilon_n$, hence

$$h_n\to 0\ \text{ in }\ W_n^{1,p}(Z)\ \text{ as }\ n\to+\infty.$$

Therefore, $\widehat{h}=0$ and so for $n\ge 1$ large we have

$$\|h_n\|_{C_n^1(\overline{Z})}\le\rho_1$$

and from this we infer that

$$\varphi_0(x_0)\le\varphi_0(x_0+h_n),$$

which contradicts (15). □

The next proposition underlines the significance of the nonuniform nonresonance condition at $+\infty$ (see the first part of hypothesis $H(j)$ (4)) and will allow the use of variational arguments on a suitably truncated energy functional.

Lemma 3.1 *If $\theta \in L^\infty(Z)$, $\theta(z) \le 0$, a.e. on Z and the inequality is strict on a set of positive measure, then there exists $\xi_0 > 0$ such that*

$$\Psi(x) = \|Dx\|_p^p - \int_Z \theta(z)|x(z)|^p \mathrm{d}z \ge \xi_0 \|x\|^p \quad \textit{for all } x \in W^{1,p}(Z).$$

Proof Evidently $\Psi \ge 0$. Suppose that the proposition is not true. Since Ψ is p-positively homogeneous, we can find $\{x_n\} \subseteq W^{1,p}(Z)$ such that $\|x_n\| = 1$ and $\Psi(x_n) \downarrow 0$. By passing to a suitable subsequence if necessary, we may assume that

$$x_n \rightharpoonup x \quad \text{in } W^{1,p}(Z), \quad x_n \to x \quad \text{in } L^p(Z), \quad x_n(z) \to x(z) \quad \text{a.e. on } Z$$

and

$$|x_n(z)| \le k(z) \quad \text{for a.a. } z \in Z, \text{ all } n \ge 1 \text{ with } k \in L^p(Z)_+ .$$

Then we have

$$\|Dx\|_p^p \le \liminf_{n \to +\infty} \|Dx_n\|_p^p$$

and from the dominated convergence theorem

$$\int_Z \theta(z)|x_n(z)|^p \mathrm{d}z \to \int_Z \theta(z)|x(z)|^p \mathrm{d}z.$$

Therefore

$$\Psi(x) \le \liminf_{n \to +\infty} \Psi(x_n),$$

and this implies that

$$\|Dx\|_p^p \le \int_Z \theta(z)|x(z)|^p \mathrm{d}z \le 0, \tag{21}$$

so $\|Dx\|_p = 0$ and

$$x \equiv c \in \mathbb{R} .$$

If $c = 0$, then $\|Dx_n\|_p \to 0$ and so $x_n \to 0$ in $W^{1,p}(Z)$, a contradiction to the fact that $\|x_n\| = 1$ for all $n \ge 1$.
If $c \ne 0$, then from the first inequality in (21) and the assumptions on θ, we have

$$0 \le |c|^p \int_Z \theta(z) \mathrm{d}z < 0,$$

again a contradiction. This completes the proof of the proposition. □

Now we are ready to produce the first positive solution of problem (1). For this purpose we introduce φ: $W_n^{1,p}(Z) \to \mathbb{R}$, the Euler functional for problem (1), defined by

$$\varphi(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z j(z, x(z)) \mathrm{d}z \quad \text{for all } x \in W_n^{1,p}(Z).$$

We know that φ is Lipschitz continuous on bounded sets, hence locally Lipschitz (see Clarke [10], p. 83).

Proposition 3.2 *If hypotheses $H(j)$ hold, then problem (1) has a solution $x_0 \in int\, C_+$ which is a local minimizer of φ.*

Proof By virtue of hypotheses $H(j)$ (4), given $\varepsilon > 0$, we can find $M = M(\varepsilon) > 0$ such that

$$u \leq (\theta(z) + \varepsilon)\, x^{p-1} \quad \text{for a.a. } z \in Z, \text{ all } x \geq M \text{ and all } u \in \partial j(z,x). \tag{22}$$

On the other hand, from hypotheses $H(j)$ (3), corresponding to M we can find $a_\varepsilon \in L^\infty(Z)_+$ such that

$$u \leq a_\varepsilon(z) \quad \text{for a.a. } z \in Z, \text{ all } 0 \leq x < M \text{ and all } u \in \partial j(z,x)\,. \tag{23}$$

From (22) and (23) and since $\theta \leq 0$, we obtain

$$u \leq (\theta(z) + \varepsilon)\, x^{p-1} + \widehat{a}_\varepsilon(z) \tag{24}$$

for a.a. $z \in Z$, all $x \geq 0$, all $u \in \partial j(z,x)$, with $\widehat{a}_\varepsilon \in L^\infty(Z)_+$. Because of hypotheses $H(j)$ (1), (2) and Rademacher's theorem, we can find $D \subseteq Z$ a Lebesgue-null set such that for all $z \in Z \backslash D$, the function $r \to j(z,r)$ is almost everywhere differentiable on $\mathbb{R}$ and at a differentiability point $r \in \mathbb{R}$, we have $\frac{\mathrm{d}}{\mathrm{d}r} j(z,r) \in \partial j(z,r)$ (see Clarke [11], p. 32). So using (24), we have

$$\frac{\mathrm{d}}{\mathrm{d}r} j(z,r) \leq (\theta(z) + \varepsilon)\, r^{p-1} + \widehat{a}_\varepsilon(z) \quad \text{for all } z \in Z \backslash D \text{ and a.a. } r \geq 0.$$

We integrate this inequality over the interval $[0,x]$, $x \geq 0$ and obtain

$$j(z,x) \leq \frac{1}{p}\, (\theta(z) + \varepsilon)\, x^p + \widehat{a}_\varepsilon(z) x \quad \text{for all } z \in Z \backslash D \text{ and all } x \geq 0. \tag{25}$$

Then, taking into account (25) and Proposition 3.1, for every $x \in W_+$, we have

$$\begin{aligned} \varphi(x) &= \frac{1}{p}\|Dx\|_p^p - \int_Z j(z,x(z))\mathrm{d}z \geq \frac{1}{p}\|Dx\|_p^p - \frac{1}{p}\int_Z \theta |x|^p \mathrm{d}z \\ &\quad - \frac{\varepsilon}{p}\|x\|^p - c_1\|x\| \geq \frac{\xi_0 - \varepsilon}{p}\|x\|^p - c_1\|x\|\,. \end{aligned} \tag{26}$$

If we choose $0 < \varepsilon < \xi_0$, from (26) we see that $\varphi_{|W_+}$ is coercive. Moreover, exploiting the compact embedding of $W_n^{1,p}(Z)$ into $L^p(Z)$, we can easily verify that φ is weakly lower semicontinuous. Therefore by the Weierstrass theorem, we can find $x_0 \in W_+$ such that

$$\varphi(x_0) = \inf_{W_+} \varphi = m_+.$$

From hypotheses $H(j)$ (6) and for $c_0 \in W_+ \setminus \{0\}$, we have

$$\varphi(c_0) = -\int_Z j(z,c_0)\mathrm{d}z < 0 = \varphi(0),$$

so

$$\varphi(x_0) = m_+ < 0 = \varphi(0), \quad \text{i.e., } x_0 \neq 0.$$

From the optimality condition of Clarke [11], p. 52, we have

$$0 \in \partial\varphi(x_0) + N_{W_+}(x_0), \tag{27}$$

with $N_{W_+}(x_0)$ being the normal cone to W_+ at x_0, i.e.,

$$N_{W_+}(x_0) = \{v^* \in W_n^{1,p}(Z)^* : \langle v^*, v - x_0\rangle \le 0 \quad \text{for all } v \in W_+\}. \tag{28}$$

From (27) we see that we can find $x^* \in \partial\varphi(x_0)$ such that $-x^* \in N_{W_+}(x_0)$. We know that (see the definition of φ)

$$x^* = A(x_0) - u_0,$$

with $u_0 \in L^{p'}(Z)$ $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$, $u_0(z) \in \partial j(z, x_0(z))$ a.e. on Z. Therefore

$$-A(x_0) + u_0 \in N_{W_+}(x_0)$$

and hence

$$0 \le \langle A(x_0) - u_0, v - x_0\rangle \quad \text{for all } v \in W_+ . \tag{29}$$

Given $\varepsilon > 0$ and $h \in W_n^{1,p}(Z)$, we set

$$v = (x_0 + \varepsilon h)^+ = (x_0 + \varepsilon h) + (x_0 + \varepsilon h)^- \in W_+ .$$

Recall that for any $w \in W_n^{1,p}(Z)$, both $w^+ = \max\{w, 0\}$ and $w^- = \max\{-w, 0\}$ belong to $W_n^{1,p}(Z)$. Using this $v \in W_+$ as test function in (29), we obtain

$$-\langle x^*, (x_0 + \varepsilon h)^-\rangle \le \varepsilon\langle x^*, h\rangle . \tag{30}$$

We set $Z_\varepsilon^- = \{z \in Z : (x_0 + \varepsilon h)(z) < 0\}$. We know that

$$D\left[(x_0 + \varepsilon h)^-\right](z) = \begin{cases} -D(x_0 + \varepsilon h)(z), & \text{if } z \in Z_\varepsilon^-, \\ 0, & \text{otherwise.} \end{cases} \tag{31}$$

Then we have

$$\begin{aligned} -\langle x^*, (x_0 + \varepsilon h)^-\rangle &= -\langle A(x_0), (x_0 + \varepsilon h)^-\rangle + \int_Z u_0(x_0 + \varepsilon h)^- \mathrm{d}z \\ &= -\int_Z |Dx_0|^{p-2} \left(Dx_0, D(x_0 + \varepsilon h)^-\right)_{\mathbb{R}^N} \mathrm{d}z + \int_Z u_0(x_0 + \varepsilon h)^- \mathrm{d}z. \end{aligned} \tag{32}$$

Next we estimate both integrals in the right hand side of (32). Because of (31)

$$\begin{aligned} -\int_Z |Dx_0|^{p-2} \left(Dx_0, D(x_0 + \varepsilon h)^-\right)_{\mathbb{R}^N} \mathrm{d}z &= \int_{Z_\varepsilon^-} |Dx_0|^{p-2} \left(Dx_0, D(x_0 + \varepsilon h)\right)_{\mathbb{R}^N} \mathrm{d}z \\ &\ge \varepsilon \int_{Z_\varepsilon^-} |Dx_0|^{p-2} \left(Dx_0, Dh\right)_{\mathbb{R}^N} \mathrm{d}z . \end{aligned} \tag{33}$$

Also (recall that $x_0 \in W_+$)

$$\begin{aligned} \int_Z u_0(x_0 + \varepsilon h)^- \mathrm{d}z &= -\int_{Z_\varepsilon^-} u_0(x_0 + \varepsilon h)\mathrm{d}z \\ &= -\varepsilon \int_{Z_\varepsilon^- \cap \{x_0 = 0\}} u_0 h \, \mathrm{d}z - \int_{Z_\varepsilon^- \cap \{x_0 > 0\}} u_0(x_0 + \varepsilon h)\mathrm{d}z . \end{aligned} \tag{34}$$

As $\partial j(z, 0) \subseteq \mathbb{R}_+$, then $u_0(z) \ge 0$ a.e. on $\{x_0 = 0\}$, while $h(z) < 0$ on $Z_\varepsilon^- \cap \{x_0 = 0\}$. Hence

$$-\varepsilon \int_{Z_\varepsilon^- \cap \{x_0 = 0\}} u_0 h \, \mathrm{d}z \ge 0. \tag{35}$$

Let $M_0 > 0$ be as in hypothesis $H(j)$ (6) and a_{M_0} the function of $L^\infty(Z)_+$ corresponding to the choice $r = M_0$ in hypothesis $H(j)$ (3). Then, bearing in mind both $H(j)$ (3) and (6)

$$\begin{aligned}
&-\int_{Z_\varepsilon^- \cap\{x_0>0\}} u_0(x_0+\varepsilon h)\mathrm{d}z \\
&= -\int_{Z_\varepsilon^- \cap\{0<x_0<M_0\}} u_0(x_0+\varepsilon h)\mathrm{d}z - \int_{Z_\varepsilon^- \cap\{x_0\ge M_0\}} u_0(x_0+\varepsilon h)\mathrm{d}z \\
&\ge -\int_{Z_\varepsilon^- \cap\{0<x_0<M_0\}} a_{M_0}(x_0+\varepsilon h)\mathrm{d}z - \int_{Z_\varepsilon^- \cap\{x_0\ge M_0\}} u_0(x_0+\varepsilon h)\mathrm{d}z \\
&\ge -\varepsilon \int_{Z_\varepsilon^- \cap\{x_0\ge M_0\}} u_0 h\,\mathrm{d}z\,.
\end{aligned} \tag{36}$$

We use (35) and (36) in (34). Hence

$$\int_Z u_0(x_0+\varepsilon h)^-\mathrm{d}z \ge -\varepsilon \int_{Z_\varepsilon^- \cap\{x_0\ge M_0\}} u_0 h\,\mathrm{d}z\,. \tag{37}$$

Then we return to (32) and plug-in (33) and (37). We obtain

$$-\langle x^*, (x_0+\varepsilon h)^-\rangle \ge \varepsilon \int_{Z_\varepsilon^-} |Dx_0|^{p-2}\,(Dx_0, Dh)_{\mathbb{R}^N}\,\mathrm{d}z - \varepsilon \int_{Z_\varepsilon^- \cap\{x_0\ge M_0\}} u_0 h\,\mathrm{d}z$$

and finally, (30) yields

$$\langle x^*, h\rangle \ge \int_{Z_\varepsilon^-} |Dx_0|^{p-2}\,(Dx_0, Dh)_{\mathbb{R}^N}\,\mathrm{d}z - \int_{Z_\varepsilon^- \cap\{x_0\ge M_0\}} u_0 h\,\mathrm{d}z\,.$$

From Stampacchia's theorem, we know that $Dx_0(z) = 0$ a.e. on $\{x_0 = 0\}$. Hence

$$\langle x^*, h\rangle \ge \int_{Z_\varepsilon^- \cap\{x_0>0\}} |Dx_0|^{p-2}\,(Dx_0, Dh)_{\mathbb{R}^N}\,\mathrm{d}z - \int_{Z_\varepsilon^- \cap\{x_0\ge M_0\}} u_0 h\,\mathrm{d}z\,. \tag{38}$$

If by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$, we see that

$$\left|Z_\varepsilon^- \cap \{x_0 > 0\}\right|_N \to 0 \quad \text{as } \varepsilon \downarrow 0\,.$$

So if in (38), we pass to the limit as $\varepsilon \downarrow 0$, we obtain

$$0 \le \langle x^*, h\rangle\,. \tag{39}$$

But $h \in W_n^{1,p}(Z)$ was arbitrary. So from (39) it follows

$$x^* = 0,$$

that is,

$$A(x_0) = u_0\,. \tag{40}$$

From (40), arguing as in the proof of Proposition 3.1, using the nonlinear Green's identity and nonlinear regularity theory, we obtain that $x_0 \in C_+$ and it solves problem (1). So

$$-\mathrm{div}\left(|Dx_0(z)|^{p-2}Dx_0(z)\right) = u_0(z) \quad \text{a.e. on } Z\,.$$

Taking into account hypothesis $H(j)$ (6) we deduce

$$\operatorname{div}\left(|Dx_0(z)|^{p-2}Dx_0(z)\right) \leq cx_0(z)^{p-1} \text{ a.e. on } Z\,.$$

Invoking the maximum principle of Vazquez [29], we obtain

$$x_0 \in \operatorname{int} C_+\,.$$

Therefore, we can find $\rho_1 > 0$ such that $x_0+\overline{B}_{\rho_1}^{C_n^1} = x_0+\left\{h \in C_n^1(\overline{Z}) : \|h\|_{C_n^1(\overline{Z})} \leq \rho_1\right\} \subseteq C_+ \subseteq W_+$. Hence, recalling that x_0 is a minimizer of φ on W_+,

$$\varphi(x_0) \leq \varphi(x_0+h) \quad \text{for all } h \in C_n^1(\overline{Z}), \quad \|h\|_{C_n^1(\overline{Z})} \leq \rho_1\,.$$

Invoking Proposition 3.1, we can find $\rho_2 > 0$ such that

$$\varphi(x_0) \leq \varphi(x_0+h) \quad \text{for all } h \in W_n^{1,p}(Z), \quad \|h\| \leq \rho_2,$$

that is, x_0 is a local $W_n^{1,p}$-minimizer of φ. □

4 Existence of a second positive solution

In this section, we produce a second positive solution for problem (1). To do this, we employ a degree theoretic approach based on the degree map $\widehat{d}$ introduced in Sect. 2.

We start with some auxiliary results which pave the way for the implementation of the degree theoretic methods. So let $A\colon W_n^{1,p}(Z) \to W_n^{1,p}(Z)^*$ be the nonlinear operator introduced in the proof of Proposition 3.1 and defined by

$$\langle A(x), y\rangle = \int_Z |Dx|^{p-2}\,(Dx, Dy)_{\mathbb{R}^N}\,\mathrm{d}z \quad \text{for all } x, y \in W^{1,p}(Z). \tag{41}$$

Proposition 4.1 *A: $W_n^{1,p}(Z) \to W_n^{1,p}(Z)^*$ is a bounded, continuous, $(S)_+$-operator.*

Proof From (41), we easily see that A is bounded, continuous, monotone, hence maximal monotone. Let $x_n \rightharpoonup x$ in $W_n^{1,p}(Z)$ and assume that $\limsup_{n\to+\infty}\langle A(x_n), x_n - x\rangle \leq 0$. Since A is maximal monotone, it is generalized pseudomonotone (see Gasinski and Papageorgiou [16], p. 330) and so

$$\|Dx_n\|_p^p = \langle A(x_n), x_n\rangle \to \langle A(x), x\rangle = \|Dx\|_p^p\,.$$

Also we have $Dx_n \rightharpoonup Dx$ in $L^p(Z, \mathbb{R}^N)$. Since $L^p(Z, \mathbb{R}^N)$ is uniformly convex, it has the Kadec–Klee property. Therefore, it follows that

$$Dx_n \to Dx \quad \text{in } L^p(Z, \mathbb{R}^N) \text{ as } n \to +\infty. \tag{42}$$

Also from the compact embedding of $W_n^{1,p}(Z)$ into $L^p(Z)$, we also have

$$x_n \to x \quad \text{in } L^p(Z) \text{ as } n \to +\infty. \tag{43}$$

From (42) and (43) we conclude that $x_n \to x$ in $W_n^{1,p}(Z)$, which proves that A is an $(S)_+$-operator. □

Let $\widehat{N}$: $L^p(Z) \to 2^{L^{p'}(Z)}(\frac{1}{p}+\frac{1}{p'}=1)$ be the multivalued map defined by

$$\widehat{N}(x) = \{u \in L^{p'}(Z) : u(z) \in \partial j(z, x(z)) \text{ a.e. on } Z\}.$$

This is the multivalued Nemytskii operator corresponding to the generalized subdifferential multifunction $x \to \partial j(z, x)$. From Proposition 3 of Aizicovici et al. [2], we know that $\widehat{N}$ has nonempty, weakly compact and convex values and it is upper semicontinuous from $L^p(Z)$ with the norm topology into $L^{p'}(Z)$ with the weak topology. Since $W_n^{1,p}(Z)$ is embedded compactly in $L^p(Z)$ and so does $L^{p'}(Z)$ into $W_n^{1,p}(Z)^*$, we deduce that:

Proposition 4.2 *If hypothesis $H(j)$ hold, then $N = \widehat{N}_{|W_n^{1,p}(Z)}$: $W_n^{1,p}(Z) \to 2^{W_n^{1,p}(Z)^*} \setminus \{\emptyset\}$ is a multifunction of class (P).*

In the next proposition, we examine the monotonicity properties of the map $m \to \widehat{\lambda}_1(m)$. A similar result for the second "Dirichlet" eigenvalue, was proved by Anane and Tsouli [3].

Proposition 4.3 *If $m, m' \in L^\infty(Z)_+$, $m \neq 0$ and $m(z) < m'(z)$ for a.a. $z \in Z$, then $\widehat{\lambda}_1(m') < \widehat{\lambda}_1(m)$.*

Proof Let $u_1 \in C_n^1(\overline{Z})$ be an eigenfunction corresponding to the eigenvalue $\widehat{\lambda}_1(m')$. From Lê [22] we know that u_1 changes sign and it has two nodal domains $Z_+ = \{u_1 > 0\}$ and $Z_- = \{u_1 < 0\}$. We set

$$w_\pm(z) = \begin{cases} \dfrac{u_1(z)}{\left(\int_{Z_\pm} m'|u_1|^p dz\right)^{1/p}}, & \text{if } z \in Z_\pm, \\ 0 & \text{otherwise}. \end{cases}$$

Clearly $w_\pm \in C_n^1(\overline{Z})$ and the two functions are linearly independent. So, if $Y = \operatorname{span}\{w_+, w_-\}$, then $\dim Y = 2$ and for $w \in Y$, we have $w = t_1 w_+ + t_2 w_-$ with $t_1, t_2 \in \mathbb{R}$. Note that

$$w \to \varphi_{m'}(w)^{1/p} = \left(\int_Z m'|w|^p \mathrm{d}z\right)^{1/p} = \left(|t_1|^p + |t_2|^p\right)^{1/p}$$

(see Sect. 2), is an equivalent norm on Y and so if we set

$$S_1 = \left\{w \in Y : \varphi_{m'}(w) = \frac{1}{\widehat{\lambda}_1(m') + 1}\right\},$$

then S_1 is homeomorphic to the unit sphere in $\mathbb{R}^2$. Therefore, if by $\gamma(\cdot)$ we denote the Krasnoselskii genus (see for example Gasinski and Papageorgiou [16], p. 680), then

$$\gamma(S_1) = 2.$$

From the definition of $\widehat{\lambda}_1(m)$ (see (4)), we have

$$\psi_{m'}(w_\pm) = \left(\widehat{\lambda}_1(m') + 1\right)\varphi_{m'}(w_\pm),$$

hence

$$\psi_{m'}(w) = \left(\widehat{\lambda}_1(m') + 1\right)\left(|t_1|^p + |t_2|^p\right) = \left(\widehat{\lambda}_1(m') + 1\right)\varphi_{m'}(w) = 1 \text{ for all } w \in S_1,$$

that is, $S_1 \subseteq S(\psi_{m'})$ (recall that $S(\psi_{m'}) = \psi_{m'}^{-1}(1)$). Using once again (4), we have

$$\inf_{w\in S_1} \varphi_{m'}(w) = \frac{1}{\widehat{\lambda}_1(m')+1}\,. \tag{44}$$

But note that for $w \in Y$, we have

$$\varphi_m(w) < \varphi_{m'}(w);$$

as S_1 is compact, from (44) we deduce

$$\frac{1}{\widehat{\lambda}_1(m)+1} < \frac{1}{\widehat{\lambda}_1(m')+1}$$

and finally

$$\widehat{\lambda}_1(m') < \widehat{\lambda}_1(m)\,.$$

□

Propositions 4.1 and 4.2, permit us to compute the $\widehat{d}$-degree of the nonlinear multivalued operator $x \to A(x) - N(x)$ for large balls, $B_R = \{x \in W_n^{1,p}(Z) : \ \|x\| < R\}$.

Proposition 4.4 *If hypotheses $H(j)$ hold, then there exists $R_0 > 0$ such that $\widehat{d}(A - N, B_R, 0) = 0$ for all $R \geq R_0$.*

Proof Let K, $K_-\colon W_n^{1,p}(Z) \to W_n^{1,p}(Z)^*$ be the nonlinear operators defined by

$$K(x)(\cdot) = |x(\cdot)|^{p-2}x(\cdot) \quad \text{and } K_-(x)(\cdot) = \big(x^-(\cdot)\big)^{p-1} \quad \text{for all } x \in W_n^{1,p}(Z).$$

Evidently both maps are completely continuous. Also we fix $h \in L^\infty(Z)_+$ such that $0 \leq h(z) < \lambda_1$ a.e. on Z, $h \neq 0$ and $|\{z \in Z : \ h(z) > 0, \theta_1(z) > 0\}|_N > 0$. Then we choose $\theta_0 \in L^\infty(Z), \theta_0 \neq 0, \theta_0 \leq 0$ a.e. on Z such that $0 \leq h(z)+\theta_0(z), 0 \leq \theta_1(z)+\theta_0(z)$ a.e. on Z, $h+\theta_0 \neq 0$, $h+\theta_1 \neq 0$. We consider the homotopy $H_1 : [0,1] \times W_n^{1,p}(Z) \to 2^{W_n^{1,p}(Z)^*} \setminus \{\emptyset\}$ defined by

$$H_1(t,x) = A(x) - \theta_0 K(x) - tN(x) + (1-t)hK_-(x).$$

This is an admissible homotopy.

Claim I *There exists $\widehat{R}_0 > 0$ such that $0 \notin H_1(t,x)$ for all $t \in [0,1]$ and all $\|x\| = R \geq \widehat{R}_0$. We argue indirectly. So suppose the Claim is not true. Then we can find $\{t_n\}_{n\geq 1} \subseteq [0,1]$, $\{x_n\}_{n\geq 1} \subseteq W_n^{1,p}(Z)$ and $\{u_n\} \in N(x_n)$ such that*

$$t_n \to t,\ \|x_n\| \to +\infty \quad \textit{and } A(x_n) - \theta_0 K(x_n) = t_n u_n - (1-t_n)hK_-(x_n). \tag{45}$$

On the equation in (45), we act with the test function $x_n^+ \in W_n^{1,p}(Z)$ and, owing to Proposition 3.1, we obtain

$$\xi_0\|x_n^+\|^p \leq \|Dx_n^+\|_p^p - \int_Z \theta_0|x_n^+|^p \mathrm{d}z = t_n \int_Z u_n x_n^+ \mathrm{d}z\,. \tag{46}$$

Suppose that $\{x_n^+\}_{n\geq 1}$ is unbounded. We may assume that $\|x_n^+\| \to +\infty$. We set $y_n = \frac{x_n^+}{\|x_n^+\|}$, $n \geq 1$. By passing to a suitable subsequence if necessary, we may assume that

$$y_n \rightharpoonup y \quad \text{in } W_n^{1,p}(Z), \quad y_n \to y \quad \text{in } L^p(Z), \quad y_n(z) \to y(z) \quad \text{a.e. on } Z$$

and

$$|y_n(z)| \leq k(z) \text{ a.e. on } Z, \text{ for all } n \geq 1, \text{ with } k \in L^p(Z)_+ .$$

Let $\widehat{u}_n(z) = \chi_{\{x_n>0\}}(z)u_n(z)$. Then

$$u_n(z)x_n^+(z) = \widehat{u}_n(z)x_n^+(z) .$$

If we use this in (46), then we obtain

$$\xi_0 \|y_n\|^p \leq t_n \int_Z \frac{\widehat{u}_n}{\|x_n^+\|^{p-1}} y_n \mathrm{d}z . \tag{47}$$

From hypotheses $H(j)$ (4), (6), we see that given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$-cx^{p-1} - c_\varepsilon \leq u \leq \varepsilon x^{p-1} + c_\varepsilon \quad \text{for a.a. } z \in Z, \quad \text{all } x \geq 0, \quad \text{all } u \in \partial j(z,x);$$

from this inequalities immediately follows that

$$-cy_n(z)^{p-1} - \frac{c_\varepsilon}{\|x_n^+\|^{p-1}} \leq \frac{\widehat{u}_n(z)}{\|x_n^+\|^{p-1}} \leq \varepsilon y_n(z)^{p-1} + \frac{c_\varepsilon}{\|x_n^+\|^{p-1}} \tag{48}$$

for a.a. $z \in Z$, all $n \geq 1$, namely

$$\left\{ \frac{\widehat{u}_n}{\|x_n^+\|^{p-1}} \right\}_{n\geq 1} \subseteq L^{p'}(Z) \quad \text{is bounded} .$$

So we may assume that

$$\frac{\widehat{u}_n}{\|x_n^+\|^{p-1}} \rightharpoonup \beta \quad \text{in } L^{p'}(Z) \text{ as } n \to +\infty .$$

From (48), Mazur's Lemma and because $\varepsilon > 0$ was arbitrary, we deduce that $\beta \leq 0$. Hence if we pass to the limit as $n \to +\infty$ in (47), we obtain

$$y_n \to 0 \quad \text{in } W_n^{1,p}(Z),$$

a contradiction to the fact that $\|y_n\| = 1$ for all $n \geq 1$. This proves that $\{x_n^+\}_{n\geq 1} \subseteq W_n^{1,p}(Z)$ is bounded. Since $\|x_n\| \to +\infty$, we must have $\|x_n^-\| \to +\infty$. Now we set $\overline{y}_n = \frac{x_n^-}{\|x_n^-\|}$, $n \geq 1$. As before we may assume that

$$\overline{y}_n \rightharpoonup \overline{y} \text{ in } W_n^{1,p}(Z), \quad \overline{y}_n \to \overline{y} \text{ in } L^p(Z), \quad \overline{y}_n(z) \to \overline{y}(z) \text{ a.e. on } Z$$

and

$$|\overline{y}_n(z)| \leq k(z) \text{ a.e. on } Z \quad \text{for all } n \geq 1 \quad \text{with } k \in L^p(Z)_+ .$$

Note that $A(x_n) = A(x_n^+) - A(x_n^-)$, $K(x_n) = K(x_n^+) - K(x_n^-)$. So from (45), we have

$$A(x_n^+) - A(x_n^-) - \theta_0 K(x_n^+) - \theta_0 K(x_n^-) = t_n u_n - (1-t_n)hK_-(x_n) .$$

Dividing with $\|x_n^-\|^{p-1}$, we have

$$\frac{1}{\|x_n^-\|^{p-1}} A(x_n^+) - A(\overline{y}_n) - \theta_0 \frac{K(x_n^+)}{\|x_n^-\|^{p-1}} - \theta_0 K(\overline{y}_n) = t_n \frac{u_n}{\|x_n^-\|^{p-1}} - (1-t_n)hK(\overline{y}_n), \tag{49}$$

acting with the test function $\overline{y}_n - \overline{y} \in W_n^{1,p}(Z)$ we obtain

$$\begin{aligned} &\frac{1}{\|x_n^-\|^{p-1}} \langle A(x_n^+), \overline{y}_n - \overline{y}\rangle - \langle A(\overline{y}_n), \overline{y}_n - \overline{y}\rangle \\ &\quad - \int_Z \theta_0 \frac{(x_n^+)^{p-1}}{\|x_n^-\|^{p-1}} (\overline{y}_n - \overline{y}) dz + \int_Z \theta_0 |\overline{y}_n|^{p-2} \overline{y}_n (\overline{y}_n - \overline{y}) dz \\ &\quad = t_n \int_Z \frac{u_n}{\|x_n^-\|^{p-1}} (\overline{y}_n - \overline{y}) dz - (1-t_n) \int_Z h |\overline{y}_n|^{p-2} \overline{y}_n (\overline{y}_n - \overline{y}) dz\,. \end{aligned} \tag{50}$$

Since the operators A and K are bounded and $\{x_n^+\}_{n\geq 1}$ is bounded, we have

$$\frac{1}{\|x_n^-\|^{p-1}} \langle A(x_n^+), \overline{y}_n - \overline{y}\rangle \to 0 \quad \text{and} \quad \int_Z \theta_0 \frac{(x_n^+)^{p-1}}{\|x_n^-\|^{p-1}} (\overline{y}_n - \overline{y}) dz \to 0 \quad \text{as } n \to +\infty. \tag{51}$$

Clearly we have

$$\int_Z h |\overline{y}_n|^{p-1} (\overline{y}_n - \overline{y}) dz \to 0 \quad \text{and} \quad \int_Z \theta_0 |\overline{y}_n|^{p-2} \overline{y}_n (\overline{y}_n - \overline{y}) dz \to 0 \text{ as } n \to +\infty. \tag{52}$$

Moreover, if we set

$$\widehat{u}_n(z) = \chi_{\{x_n \geq 0\}}(z) u_n(z) \quad \text{and} \quad \overline{u}_n(z) = \chi_{\{x_n < 0\}}(z) u_n(z)$$

then

$$\int_Z \frac{u_n}{\|x_n^-\|^{p-1}} (\overline{y}_n - \overline{y}) dz = \int_Z \frac{\widehat{u}_n}{\|x_n^-\|^{p-1}} (\overline{y}_n - \overline{y}) dz + \int_Z \frac{\overline{u}_n}{\|x_n^-\|^{p-1}} (\overline{y}_n - \overline{y}) dz. \tag{53}$$

From the boundedness of $\{x_n^+\}_{n\geq 1}$, we see that $\{\widehat{u}_n\}_{n\geq 1} \subseteq L^{p'}(Z)$ is bounded and so

$$\int_Z \frac{\widehat{u}_n}{\|x_n^-\|^{p-1}} (\overline{y}_n - \overline{y}) dz \to 0 \quad \text{as } n \to +\infty. \tag{54}$$

Furthermore, hypotheses $H(j)$ (3)–(6), imply that

$$|u| \leq c_2 |x|^{p-1} \quad \text{for a.a. } z \in Z, \quad \text{all } x \in \mathbb{R} \quad \text{all } u \in \partial j(z,x) \quad \text{with } c_2 > 0,$$

so

$$\frac{|\overline{u}_n(z)|}{\|x_n^-\|^{p-1}} \leq c_2 \frac{x_n^-(z)^{p-1}}{\|x_n^-\|^{p-1}} + c_2 \overline{y}_n(z)^{p-1} \text{ a.e. on } Z, \quad \text{for all } n \geq 1, \tag{55}$$

hence

$$\left\{ \frac{\overline{u}_n}{\|x_n^-\|^{p-1}} \right\}_{n\geq 1} \subseteq L^{p'}(Z) \text{ is bounded}$$

and from this we deduce that

$$\int_Z \frac{\overline{u}_n}{\|x_n^-\|^{p-1}} (\overline{y}_n - \overline{y}) dz \to 0 \quad \text{as } n \to +\infty. \tag{56}$$

Using (54) and (56) in (53), we see that

$$\int_Z \frac{u_n}{\|x_n^-\|^{p-1}} (\overline{y}_n - \overline{y}) dz \to 0 \quad \text{as } n \to +\infty. \tag{57}$$

So, if we return to (50), pass to the limit as $n \to +\infty$ and use (51), (52), and (57), we have

$$\lim_{n\to+\infty} \langle A(\overline{y}_n), \overline{y}_n - \overline{y}\rangle = 0.$$

Since A is an $(S)_+$-operator (see Proposition 4.1), it follows that

$$\overline{y}_n \to \overline{y} \ \text{ in } \ W_n^{1,p}(Z), \qquad \|\overline{y}\| = 1\,.$$

Because of the boundedness of $\left\{\frac{\overline{u}_n}{\|x_n^-\|^{p-1}}\right\}_{n\geq 1}$ in $L^{p'}(Z)$, we may assume that

$$\frac{\overline{u}_n}{\|x_n^-\|^{p-1}} \rightharpoonup \gamma \ \text{ in } \ L^{p'}(Z) \quad \text{as } \ n \to +\infty.$$

For every $\varepsilon > 0$ and $n \geq 1$, we consider the set

$$C_{\varepsilon,n} = \left\{ z \in Z;\, x_n(z) < 0,\ \theta_1(z) - \varepsilon \leq \frac{\overline{u}_n(z)}{-x_n^-(z)^{p-1}} \leq \theta_2(z) + \varepsilon \right\}.$$

Note that $x_n^-(z) \to +\infty$ for a.a. $z \in \{\overline{y} > 0\}$. So by virtue of hypotheses $H(j)$ (4), we have

$$\chi_{C_{\varepsilon,n}} \to 1 \quad \text{a.e. on } \{\overline{y} > 0\}$$

and the dominated convergence theorem yields

$$\left\| \left(1 - \chi_{C_{\varepsilon,n}}\right) \frac{\overline{u}_n}{\|x_n^-\|^{p-1}} \right\|_{L^{p'}(\{\overline{y}>0\})} \to 0,$$

so

$$\chi_{C_{\varepsilon,n}} \frac{\overline{u}_n}{\|x_n^-\|^{p-1}} \rightharpoonup \gamma \quad \text{in } L^{p'}\left(\{\overline{y} > 0\}\right). \tag{58}$$

From the definition of the set $C_{\varepsilon,n}$, we have

$$-\chi_{C_{\varepsilon,n}}(z)\,(\theta_2(z) + \varepsilon)\,\overline{y}_n(z)^{p-1} \leq \chi_{C_{\varepsilon,n}}(z) \frac{\overline{u}_n}{\|x_n^-\|^{p-1}}$$

$$\leq -\chi_{C_{\varepsilon,n}}(z)\,(\theta_1(z) - \varepsilon)\,\overline{y}_n(z)^{p-1} \quad \text{a.e. on } Z.$$

Passing to the limit as $n \to +\infty$ and using (58) and Mazur's lemma, we obtain

$$-\left(\theta_2(z) + \varepsilon\right)\overline{y}(z)^{p-1} \leq \gamma(z) \leq -\left(\theta_1(z) - \varepsilon\right)\overline{y}(z)^{p-1} \quad \text{a.e. on } \{\overline{y} > 0\}.$$

Since $\varepsilon > 0$ was arbitrary, we let $\varepsilon \downarrow 0$ and so

$$-\theta_2(z)\overline{y}(z)^{p-1} \leq \gamma(z) \leq -\theta_1(z)\overline{y}(z)^{p-1} \quad \text{a.e. on } \{\overline{y} > 0\}. \tag{59}$$

In addition, from (55) we see that

$$\gamma(z) = 0 \quad \text{a.e. on } \{\overline{y} = 0\}. \tag{60}$$

Since $\overline{y} \geq 0$, from (60) it follows that (59) holds a.e. on Z, so we can write

$$\gamma(z) = -g(z)\overline{y}(z)^{p-1} \quad \text{a.e. on } Z,$$

with $g \in L^{\infty}(Z)_+$, $\theta_1(z) \le g(z) \le \theta_2(z)$ a.e. on Z. If we pass to the limit as $n \to +\infty$ in (49) and recalling that $y_n \to y$ in $W_n^{1,p}(Z)$, we obtain

$$-A(y) + \theta_0 K(y) = -tgK(y) - (1-t)hK(y),$$

that is,

$$A(y) - \theta_0 K(y) = \xi K(y) \quad \text{with } \xi = tg + (1-t)h \in L^{\infty}(Z)_+. \tag{61}$$

As before, from (61), we have

$$\begin{aligned} -\mathrm{div}(|Dy(z)|^{p-2}Dy(z)) &= (\xi + \theta_0)(z)|y(z)|^{p-2}y(z) && \text{a.e. on } Z, \\ \frac{\partial y}{\partial n_p} &= 0 && \text{on } \partial Z. \end{aligned} \tag{62}$$

Note that from the choice of θ_0, we have $0 \le (\xi + \theta_0)(z) < \lambda_1$ a.e. on Z, $(\xi + \theta_0) \neq 0$. Then Proposition 4.3 implies that

$$1 = \widehat{\lambda}_1(\lambda_1) < \widehat{\lambda}_1(\xi + \theta_0) \tag{63}$$

and, since $(\xi + \theta_0) \neq 0$ we have

$$\widehat{\lambda}_0(\xi + \theta_0) = 0. \tag{64}$$

From (63) and (64) it follows that 1 is not an eigenvalue of $\left(-\triangle_p, W_n^{1,p}(Z), \xi + \theta_0\right)$. This together with (62), implies that $y = 0$, a contradiction. So we have proved Claim I. Then Claim I and the homotopy invariance property of the degree map $\widehat{d}$, imply

$$\widehat{d}(A - \theta_0 K - N, B_R, 0) = d_{(S)_+}(A - \theta_0 K + hK_-, B_R, 0) \quad \text{for all } R \ge \widehat{R}_0. \tag{65}$$

Next we compute $d_{(S)_+}(A - \theta_0 K + hK_-, B_R, 0)$: to this end let $\mu \in L^{\infty}(Z)_+$, $\mu \neq 0$ and consider the $(S)_+$-homotopy H_2: $[0,1] \times W_n^{1,p}(Z) \to 2^{W_n^{1,p}(Z)^*} \setminus \{\emptyset\}$ defined by

$$H_2(t,x) = A(x) - \theta_0 K(x) + hK_-(x) + t\mu.$$

We show that for all $t \in [0,1]$ and all $x \neq 0$, we have

$$H_2(t,x) \neq 0.$$

Indeed, if this is not the case, then for some $t \in [0,1]$ and for some $x \in W_n^{1,p}(Z)$, $x \neq 0$, we have

$$A(x) - \theta_0 K(x) = -hK_-(x) - t\mu.$$

Acting with the test function x^+, since $\mu \ge 0$ using Proposition 3.1, we have

$$\xi_0 \|x^+\|^p \le 0,$$

hence

$$x \le 0, \quad x \neq 0.$$

Then

$$A(x) - \theta_0 K(x) = -hK(-x) - t\mu$$

and this equality can be written as

$$A(-x) - \theta_0 K(-x) = hK(-x) + t\mu. \tag{66}$$

From (66) we obtain

$$\{-\mathrm{div}(|D(-x)(z)|^{p-2}D(-x)(z)) = (h+\theta_0)\,(z)|-x(z)|^{p-2}(-x)(z) + t\mu(z) \quad \text{a.e. on } Z, \qquad \frac{\partial(-x)}{\partial n_p} = 0 \quad \text{on } \partial Z\,. \tag{67}$$

From nonlinear regularity theory, we have $x \in C_n^1(\overline{Z})$. Recall that from the choice of θ_0, we have that 1 is not an eigenvalue of $\left(-\triangle_p, W_n^{1,p}(Z), h+\theta_0\right)$. If $t = 0$, then from (52), we see that $x = 0$. If $0 < t \le 1$, then from Remark 2.10 of Godoy et al. [17], we have that problem (67) can not have a nontrivial positive solution, a contradiction to the fact that $-x \in C_+, x \neq 0$. So once again the homotopy invariance of the degree map $d_{(S)_+}$ implies

$$d_{(S)_+}(A - \theta_0 K + hK_-, B_R, 0) = d_{(S)_+}(A - \theta_0 K + hK_- + \mu, B_R, 0) \quad \text{for all } R > 0. \tag{68}$$

But from previous argument (in particular from Remark 2.10 of Godoy et al. [17]), we see that

$$d_{(S)_+}(A - \theta_0 K + hK_- + \mu, B_R, 0) = 0,$$

so, recalling (68) and (65) we obtain

$$\widehat{d}\,(A - \theta_0 K - N, B_R, 0) = 0 \quad \text{for all } R \ge \widehat{R}_0\,. \tag{69}$$

Finally, we consider the admissible homotopy H_3: $[0,1] \times W_n^{1,p}(Z) \to 2^{W_n^{1,p}(Z)^*} \setminus \{\emptyset\}$ defined by

$$H_3(t,x) = A(x) - N(x) - t\theta_0 K(x)\,.$$

Claim II *There exists $\overline{R}_0 \ge 0$ such that $0 \notin H_3(t,x)$ for all $t \in [0,1]$ and all $\|x\| = R \ge \overline{R}_0$.*
As before, we proceed by contradiction. So suppose we can find $\{t_n\}_{n\ge1} \subseteq [0,1]$ and $\{x_n\}_{n\ge1} \subseteq W_n^{1,p}(Z)$ and $\{u_n\}_{n\ge1}$, with $u_n \in N(x_n)$ for all $n \in \mathbb{N}$, such that

$$t_n \to t,\ \|x_n\| \to +\infty \quad \text{and} \ \ A(x_n) - t_n\theta_0 K(x_n) = u_n, \quad n \ge 1\,. \tag{70}$$

On the equation in (70) we act with the test function $x_n^+ \in W_n^{1,p}(Z)$ and, recalling that $\theta_0 \le 0$, we have

$$\|Dx_n^+\|_p^p \le \int_Z u_n x_n^+ \mathrm{d}z\,. \tag{71}$$

Suppose that $\{x_n^+\}_{n\ge1} \subseteq W_n^{1,p}(Z)$ is unbounded. We may assume that $\|x_n^+\| \to +\infty$. We set $\widehat{y}_n = \frac{x_n^+}{\|x_n^+\|}$. At least for a subsequence, we have

$$\widehat{y}_n \rightharpoonup \widehat{y} \text{ in } W_n^{1,p}(Z), \quad \widehat{y}_n \to \widehat{y} \text{ in } L^p(Z), \quad \widehat{y}_n(z) \to \widehat{y}(z) \quad \text{a.e. on } Z$$

and

$$|\widehat{y}_n(z)| \le k(z) \text{ a.e. on } Z, \quad \text{for all } n \ge 1, \quad \text{with } k \in L^p(Z)_+.$$

We divide (71) with $\|x_n^+\|$ and we obtain

$$\|D\widehat{y}_n\|_p^p \le \int_Z \frac{u_n}{\|x_n^+\|^{p-1}}\widehat{y}_n \mathrm{d}z\,.$$

If we set $\widehat{u}_n(z) = \chi_{\{x_n>0\}}(z)u_n(z)$, then

$$\|D\widehat{y}_n\|_p^p \leq \int_Z \frac{\widehat{u}_n}{\|x_n^+\|^{p-1}}\widehat{y}_n \mathrm{d}z\,. \tag{72}$$

From (48) we know that $\left\{\frac{\widehat{u}_n}{\|x_n^+\|^{p-1}}\right\}_{n\geq 1} \subseteq L^{p'}(Z)$ is bounded, so we may assume that

$$\frac{\widehat{u}_n}{\|x_n^+\|^{p-1}} \rightharpoonup \beta \quad \text{in } L^{p'}(Z) \ \text{ as } n \to +\infty\,.$$

Note that $x_n^+(z) \to +\infty$ a.e. on $\{\widehat{y} > 0\}$. So, as before, we can show that

$$\beta = \widehat{g}K(y),$$

with $\widehat{g} \in L^\infty(Z)$, $-c \leq \widehat{g}(z) \leq \theta(z)$ a.e. on Z. Passing to the limit as $n \to +\infty$ in (72), and recalling that $\widehat{g} \leq 0$ and $\widehat{y} \geq 0$, we obtain

$$\|D\widehat{y}\|_p^p \leq \int_Z \widehat{g}\,\widehat{y}^{p-1}\mathrm{d}z \leq 0, \tag{73}$$

namely

$$\widehat{y} = \widehat{\xi} \in \mathbb{R}_+.$$

If $\widehat{\xi} = 0$, then $\widehat{y}_n \to 0$ in $W_n^{1,p}(Z)$, a contradiction to the fact that $\|\widehat{y}_n\| = 1$ for all $n \geq 1$. If $\widehat{\xi} > 0$, then using the first inequality in (73), we have

$$0 \leq \widehat{\xi}^{p-1}\int_Z \widehat{g}\,\mathrm{d}z < 0,$$

again a contradiction. In this way, we have proved that $\{x_n^+\}_{n\geq 1} \subseteq W_n^{1,p}(Z)$ is bounded. Since $\|x_n\| \to +\infty$, we must have $\|x_n^-\| \to +\infty$. Now we set $\widetilde{y}_n = \frac{x_n^-}{\|x_n^-\|}$, $n \geq 1$. We can say that

$$\widetilde{y}_n \rightharpoonup \widetilde{y} \text{ in } W_n^{1,p}(Z), \quad \widetilde{y}_n \to \widetilde{y} \text{ in } L^p(Z), \quad \widetilde{y}_n(z) \to \widetilde{y}(z) \text{ a.e. on } Z$$

and

$$|\widetilde{y}_n(z)| \leq k(z) \text{ a.e. on } Z \quad \text{for all } n \geq 1, \text{ with } k \in L^p(Z)_+.$$

Note that $-x_n^-(z) \to -\infty$ a.e. on $\{\widetilde{y} > 0\}$. So, arguing as before, we obtain

$$A(\widetilde{y}) = (t\theta_0 + \widetilde{g})K(\widetilde{y}), \quad \widetilde{y} \neq 0, \quad \|\widetilde{y}\| = 1, \tag{74}$$

with $\widetilde{g} \in L^\infty(Z)_+$, $\theta_1(z) \leq \widetilde{g}(z) \leq \theta_2(z)$ a.e. on Z. From (74), we have

$$\begin{array}{ll} -\mathrm{div}(|D\widetilde{y}(z)|^{p-2}D\widetilde{y}(z)) = (t\theta_0 + \widetilde{g})\,(z)|\widetilde{y}(z)|^{p-2}\widetilde{y}(z) & \text{a.e. on } Z, \\ \frac{\partial \widetilde{y}}{\partial n_p} = 0 & \text{on } \partial Z\,. \end{array} \tag{75}$$

Since $1 = \widehat{\lambda}_1(\lambda_1) < \widehat{\lambda}_1\,(\widetilde{g} + t\theta_0)$, from (75) it follows that $\widetilde{y} = 0$, a contradiction with (74). This proves Claim II.
The homotopy invariance of the degree map $\widehat{d}$, implies

$$\widehat{d}\,(A - N, B_R, 0) = \widehat{d}\,(A - \theta_0 K - N, B_R, 0) \quad \text{for all } R \geq \overline{R}_0\,. \tag{76}$$

So if $R_0 = \max\{\widehat{R}_0, \overline{R}_0\}$, then from (69) and (76), we infer that

$$\widehat{d}\,(A - N, B_R, 0) = 0 \quad \text{for all } R \geq R_0. \qquad \square$$

Next we conduct a similar computation for small balls.

Proposition 4.5 *If hypotheses* $H(j)$ *hold, then there exists* $\rho_0 > 0$ *such that*

$$\widehat{d}\left(A - N, B_\rho, 0\right) = 1 \quad \textit{for all } 0 < \rho \leq \rho_0 .$$

Proof Fix $h \in L^\infty(Z)$, $h(z) \leq 0$ a.e. on Z with strict inequality on a set of positive measure and consider the admissible homotopy H_4: $[0,1] \times W_n^{1,p}(Z) \to 2^{W_n^{1,p}(Z)^*} \setminus \{\emptyset\}$ defined by

$$H_4(t,x) = A(x) - tN(x) - (1-t)hK(x) .$$

Claim There exists $\rho_0 > 0$ such that $0 \notin H_4(t,x)$ for all $t \in [0,1]$ and all $0 < \|x\| \leq \rho_0$. As in the proof of Proposition 4.4, we argue by contradiction so suppose we can find $\{t_n\}_{n\geq 1} \subseteq [0,1]$, $\{x_n\}_{n\geq 1} \subseteq W_n^{1,p}(Z)$, $\{u_n\}_{n\geq 1}$, with $u_n \in N(x_n)$ such that

$$t_n \to t, \ \|x_n\| \to 0 \quad \text{and } A(x_n) = t_n u_n + (1-t_n)hK(x_n), \quad n \geq 1 . \tag{77}$$

We set $y_n = \frac{x_n}{\|x_n\|}$, $n \geq 1$. We may assume that

$$y_n \rightharpoonup y \text{ in } W_n^{1,p}(Z), \quad y_n \to y \text{ in } L^p(Z), \quad y_n(z) \to y(z) \text{ a.e. on } Z$$

and

$$|y_n(z)| \leq k(z) \text{ a.e. on } Z, \quad \text{for all } n \geq 1 \quad \text{with } k \in L^p(Z)_+.$$

We divide the Eq. 77 with $\|x_n\|^{p-1}$ and we obtain

$$A(y_n) = t_n \frac{u_n}{\|x_n\|^{p-1}} + (1-t_n)K(y_n). \tag{78}$$

From the proof of Proposition 4.4, we know that

$$|u| \leq c_2 |x|^{p-1} \text{ for a.a. } z \in Z, \quad \text{all } x \in \mathbb{R}, \text{ all } u \in \partial j(z,x), \quad \text{with } c_2 > 0.$$

It follows that $\left\{\frac{u_n}{\|x_n\|^{p-1}}\right\}_{n\geq 1} \subseteq L^{p'}(Z)$ is bounded and we may assume that

$$\frac{u_n}{\|x_n\|^{p-1}} \rightharpoonup \widehat{h}_0 \quad \text{in } L^{p'}(Z) .$$

Arguing as in the proof of Proposition 4.4, using this time Hypothesis $H(j)$ (5), we can show that

$$\widehat{h} = \widehat{g}_0 K(y),$$

with $\widehat{g}_0 \in L^\infty(Z)$, $\eta_1(z) \leq \widehat{g}_0(z) \leq \eta_2(z)$ a.e. on Z. Moreover, acting on (78) with $y_n - y \in W_n^{1,p}(Z)$, passing to the limit as $n \to +\infty$, we obtain

$$\lim_{n\to+\infty} \langle A(y_n), y_n - y\rangle = 0 .$$

Because A is an $(S)_+$-operator (see Proposition 4.3), it follows that

$$y_n \to y \text{ in } W_n^{1,p}(Z), \quad \|y\| = 1 .$$

So, if we pass to the limit as $n \to +\infty$ in (78), we have

$$A(y) = \left(t\widehat{g}_0 + (1-t)h\right) K(y) = \widehat{\xi}_0 K(y), \tag{79}$$

with $\widehat{\xi}_0 \in L^\infty(Z)$, $\eta_1(z) \le \widehat{\xi}_0(z) \le \eta_2(z)$ a.e. on Z. Acting on (79) with $y \in W_n^{1,p}(Z)$, we obtain

$$\|Dy\|_p^p = \int_Z \widehat{\xi}_0 |y|^p \mathrm{d}z \le 0 \tag{80}$$

so, since $\|y\| = 1$

$$y \equiv c \in \mathbb{R} \setminus \{\emptyset\}.$$

Then, bearing in mind that $\widehat{\xi} \le 0$, from (80) we deduce that

$$0 = |c|^p \int_Z \widehat{\xi}_0(z)\mathrm{d}z < 0,$$

a contradiction. This proves the Claim.

From the Claim and the homotopy invariance property of the degree map $\widehat{d}$ we have

$$\widehat{d}\left(A - N, B_\rho, 0\right) = d_{(S)_+}\left(A - hK, B_\rho, 0\right) \quad \text{for all } 0 < \rho \le \rho_0. \tag{81}$$

But since $h \le \eta_1$, from Drabek [12], we know that

$$d_{(S)_+}\left(A - hK, B_\rho, 0\right) = 1$$

and so

$$\widehat{d}\left(A - N, B_\rho, 0\right) = 1 \quad \text{for all } 0 < \rho \le \rho_0. \qquad \square$$

Now we are ready for the multiplicity result concerning problem (1).

Theorem 4.1 *If hypotheses $\underline{H}(j)$ hold, then there exist at least two solutions for problem (1), $x_0 \in int\, C_+$ and $\widehat{x} \in C_n^1(\overline{Z})$, $\widehat{x} \neq 0$.*

Proof The first solution $x_0 \in \operatorname{int} C_+$ was obtained in Proposition 3.2 and it is a local minimizer of φ. We may assume that x_0 is an isolated local minimizer of φ, or otherwise we have a whole sequence of solutions of (1). Therefore, we can find $r_0 > 0$ such that

$$\varphi(x_0) < \varphi(y) \quad \text{and } 0 \notin \partial\varphi(y) \text{ for all } y \in \overline{B}_{r_0}(x_0), \quad y \neq x_0. \tag{82}$$

Claim For every $0 < r < r_0$, we have

$$\inf\Big[\varphi(x) : x \in \overline{B}_{r_0}(x_0) \setminus B_r(x_0)\Big] > \varphi(x_0). \tag{83}$$

Suppose that we can find $0 < r < r_0$ such that (83) fails. Therefore, we can find $\{x_n\}_{n\ge1} \subseteq \overline{B}_{r_0}(x_0) \setminus B_r(x_0)$ such that $\varphi(x_n) \downarrow \varphi(x_0)$ as $n \to +\infty$. We may assume that

$$x_n \rightharpoonup \overline{x} \text{ in } W_n^{1,p}(Z), \quad x_n \to \overline{x} \text{ in } L^p(Z), \quad x_n(z) \to \overline{x}(z) \text{ a.e. on } Z$$

and

$$|x_n(z)| \le k(z) \text{ a.e. on } Z, \text{ for all } n \ge 1, \text{ with } k \in L^p(Z)_+.$$

We have $\overline{x} \in \overline{B}_{r_0}(x_0)$. Since φ is weakly lower semicontinous

$$\varphi(\overline{x}) \le \lim \varphi(x_n) = \varphi(x_0),$$

hence, because of (82), we infer that $\overline{x} = x_0$. The nonsmooth mean value theorem (see Clarke [11], p. 41), gives $\lambda_n \in (0, 1)$ such that

$$\varphi(x_n) - \varphi(\frac{x_n + x_0}{2}) = \left\langle u_n^*, \frac{x_n - x_0}{2} \right\rangle \tag{84}$$

with $u_n^* \in \partial\varphi\left(\lambda_n x_n + (1 - \lambda_n)\frac{x_n+x_0}{2}\right)$. We know that

$$u_n^* = A\left(\lambda_n x_n + (1 - \lambda_n)\frac{x_n + x_0}{2}\right) - u_n$$

with $u_n \in N\left(\lambda_n x_n + (1 - \lambda_n)\frac{x_n+x_0}{2}\right)$. So, if we take into account that N is of class (P) and pass to the limit as $n \to +\infty$ in (84), then we obtain

$$\limsup_{n\to+\infty}\left\langle A\left(\lambda_n x_n + (1 - \lambda_n)\frac{x_n + x_0}{2}\right), \frac{x_n - x_0}{2}\right\rangle \le 0\,,$$

that is,

$$\limsup_{n\to+\infty}\left\langle A\left(\lambda_n x_n + (1 - \lambda_n)\frac{x_n + x_0}{2}\right), \lambda_n x_n + (1 - \lambda_n)\frac{x_n + x_0}{2} - x_0\right\rangle \le 0\,. \tag{85}$$

Since A is an $(S)_+$-operator (see Proposition 4.1), from (85) we infer that

$$\lambda_n x_n + (1 - \lambda_n)\frac{x_n + x_0}{2} \to x_0 \text{ in } W_n^{1,p}(Z)\,.$$

However, note that

$$\left\|\lambda_n x_n + (1 - \lambda_n)\frac{x_n + x_0}{2} - x_0\right\| = (1 + \lambda_n)\left\|\frac{x_n - x_0}{2}\right\| \ge \frac{r}{2} > 0\,,$$

a contradiction. So (83) holds. This means that

$$\mu = \inf\left[\varphi(x) : \ x \in B_{r_0}(x_0) \setminus B_{\frac{r_0}{2}}(x_0)\right] - \varphi(x_0) > 0\,. \tag{86}$$

We set

$$V = \{x \in B_{\frac{r_0}{2}}(x_0) : \ \varphi(x) - \varphi(x_0) < \mu\}\,.$$

Evidently this is an open bounded neighborhood of x_0. We can apply Proposition 5 of Aizicovici et al. [2] with the following data:

$$x_0,\ U = B_{r_0}(x_0), \varphi_{|B_{r_0}(x_0)} - \varphi(x_0),\ \mu \text{ as in (86)},$$

$$r \in \left(0, \frac{r_0}{2}\right) \text{ such that } B_r(x_0) \subseteq V \text{ and}$$

$$0 < \lambda < \inf\left[\varphi(x) : \ x \in B_{r_0}(x_0) \setminus B_r(x_0)\right] - \varphi(x_0)\,.$$

Then we have

$$\widehat{d}\,(A - N, V, 0) = 1\,.$$

From the excision property, we have

$$\widehat{d}\,(A - N, V, 0) = \widehat{d}\,(A - N, B_r(x_0), 0)$$

and finally

$$\widehat{d}\,(A - N, B_r(x_0), 0) = 1\,. \tag{87}$$

From Propositions 4.4 and 4.5 we know that we can find $0 < \rho_0 < R_0$ such that

$$x_0 \in B_{R_0} \setminus \overline{B}_{\rho_0},$$

$$\widehat{d}\,(A - N, B_R, 0) = 0 \text{ for all } R \geq R_0 \tag{88}$$

and

$$\widehat{d}\left(A - N, B_\rho, 0\right) = 1 \text{ for all } 0 < \rho \leq \rho_0\,. \tag{89}$$

We can always assume that $B_r(x_0) \subseteq B_{R_0}$ and $B_r(x_0) \cap B_\rho = \emptyset$. Then from the domain additivity property, we have for $0 < \rho \leq \rho_0 < R_0 \leq R$:

$$\begin{aligned}\widehat{d}\,(A - N, B_R, 0) &= \widehat{d}\left(A - N, B_\rho, 0\right)\\ &\quad + \widehat{d}\,(A - N, B_r(x_0), 0) + \widehat{d}\left(A - N, B_R \setminus \overline{\left(B_r(x_0) \cup B_\rho\right)}, 0\right),\end{aligned}$$

so, bearing in mind (87), (88) and (89), we deduce

$$\widehat{d}\left(A - N, B_R \setminus \overline{\left(B_r(x_0) \cup B_\rho\right)}, 0\right) = -2\,.$$

From the solution property, it follows that there exists $\widehat{x} \in B_R \setminus \overline{\left(B_r(x_0) \cup B_\rho\right)}$ (hence $\widehat{x} \neq x_0, \widehat{x} \neq 0$) such that

$$A(\widehat{x}) = \widehat{u} \text{ with } \widehat{u} \in N(\widehat{x}),$$

so $\widehat{x}$ solves problem (1) and from the nonlinear regularity theory, we have $\widehat{x} \in C^1_n(\overline{Z})$, $\widehat{x} \neq 0$. □

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